# Matrices - Notes

## Recommended reading and other resources

* Searle S.R. & Khuri A.I. (2017) *Matrix Algebra useful for statistics*. Second edition. John Wiley & Sons
* Any ‘Matrix algebra’ book or ‘Matrix algebra summary’ in statistics books
* If all else fails, just Google “matrix algebra”…

## Basic definitions and notations

An matrix is a rectangular array of numbers with rows and columns:

, (rows) and (columns)

The **elements** of a matrix are . For example, the elements of the matrix

are , , , , and .

The **order** of a matrix is the number of rows by the number of columns i.e. . For example,

is a matrix of order , and is of order .

A **column vector** with elements, , is a matrix with only one column, i.e. an matrix.

A **row vector** with elements, , is a matrix with only one row, i.e. a matrix.

A **transposed matrix** (or) arises from the matrix by interchanging the column vectors and the row vectors, i.e. (so a column vector is converted into a row vector and vice versa). For example,

is a row vector and is a column vector,

is of order and is of order .

A **partitioned matrix** is a matrix written in terms of sub-matrices. For example,

, where , , and are sub-matrices, and

* and have the same number of rows (as do and )
* and have the same number of columns (as do and )
* partitioning is not restricted to dividing a matrix into just four sub-matrices.

For example,

can be written as where

, , and .

A **square matrix** has exactly as many rows as it has columns, i.e. the order of the matrix is . For example, is a square matrix of order .

The **main diagonal** (or **leading diagonal**) of a square matrix are the elements lying on the diagonal from top left to bottom right: , , … , . i.e. all for . For example, the diagonal elements of matrix are 1, 4 and 6.

The **trace** of a square matrix is the sum of all the diagonal elements, i.e.

For example, is a square matrix, the main diagonal of this matrix are the elements 1 and 4, so .

## Special Matrices

A **symmetric matrix** is a square matrix for which = for , i.e. for all off-diagonal elements, so that the matrix is symmetric about the diagonal, e.g.

is a symmetric matrix, and .

A **diagonal matrix** is a square matrix where all the off-diagonal elements are zero, e.g.

.

A **zero** (or **null**) **matrix** is a matrix where all the elements are zero, e.g.

.

An **identity** (or **unit**) **matrix** is a special case of a diagonal matrix having all the diagonal elements equal to 1, e.g.

.

A ‘**summing vector**’ is a vector where all elements are 1, e.g. .

A ‘**J**’ **matrix** is a matrix (not necessarily square) whose every element is 1, e.g.

.

## Basic Operations

**Addition** and **subtraction** can only take place when the matrices involved are of the same order, i.e. they have the same number of rows and columns:

**.**

For example, if and where and are both of order then

and

Rules for **addition** and **subtraction** of matrices:

**Multiplication by a scalar**  means multiplying every element of the matrix by :

**.**

For example, if and then, .

Rules for **multiplication by a scalar**:

**Multiplication by a vector**. An matrix can be multiplied by column vector yielding a column vector providing the number of columns in is equal to the number of rows in :

i.e. for .

For example, suppose and . Note that matrix is of order and column vector is of order , i.e. the number of columns in is equal to the number of rows in . Then the product is defined yielding a vector:

**Multiplication of matrices**. The product is defined only when the number of columns in is the same as the number of rows in . The elements of are given as

where and .

If is a matrix of order , and is a matrix of order , then the product will be of order . For example, if , i.e. of order and , i.e. of order, then

which is of order , as expected.

Rules for **multiplication of matrices**:

* where is the identity matrix with the same number of rows (and columns since is square) as columns in .

So if and , i.e. the identity matrix of order , then

**.**

If and , i.e. a matrix of order , then

.

## Further definitions

The **determinant** of a second order square matrix is

For example, if then .

See text books for the definition of the determinant of higher order matrices.

The **inverse** of a matrix , **,** if it exists, is a matrix whose product with is the identity matrix, i.e.  **.** Note that both and must be square.

For second order matrices: .

For example, if then .

Then the inverse of , and

.

A **singular** or **non-invertible** matrix has the property

For example, then and therefore is a singular matrix.

An **idempotent** matrix is a square matrix with the property that .

For example, if then

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An **orthogonal** matrix is a square matrix whose product with its transpose () is equal to the identity matrix, i.e. (or whose transpose is equal to its inverse ).

For example, is orthogonal since

and so that

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